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Algebraic approach to quantum field theory on a class of noncommutative curved spacetimes

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Abstract In this article we study the quantization of a free real scalar field on a class of noncommutative manifolds, obtained via formal deformation quantization using triangular Drinfel'd twists. We construct deformed quadratic action functionals and compute the corresponding equation of motion operators. The Green's operators and the fundamental solution of the deformed equation of motion are obtained in terms of formal power series. It is shown that, using the deformed fundamental solution, we can define deformed $*$-algebras of field observables, which in general depend on the spacetime deformation parameter. This dependence is absent in the special case of Killing deformations, which include in particular the Moyal-Weyl deformation of the Minkowski spacetime.

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1 Introduction

Quantum field theory (QFT) on noncommutative (NC) spacetimes is a subject of particular interest in modern mathematical and theoretical physics, see e. g. [1,2] for reviews. Even though some work on interacting QFTs on canonically deformed Euclidean or Minkowski space has been done using modified Feynman rules or other perturbative methods, the structure of free NC QFTs

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has still not been elucidated completely. For a collection of different approaches to QFT on Moyal-Weyl- or κ-deformed Minkowski spacetime see [3] and references therein. Furthermore, there are attempts to generalize QFT to projective modules [4] and to spectral geometries [5]. Depending on the approach, one finds that NC quantum fields may exhibit new features, e. g. a non-standard, so-called twisted, statistics.

In this work, we provide a generalization of the algebraic approach to QFT (see e. g. [6,7] and references therein) to the realm of formally deformed manifolds, obtained by triangular Drinfel'd twists [8]. Even if twist-deformed (pseudo-)Riemannian manifolds may obey deformed isometry properties [9], we do not include these structures in the present work and focus on the QFT on in general nonsymmetric metric backgrounds. We will see later that the NC quantum field theory is described by an (in general deformation parameter dependent) $\ast$-algebra of field observables, which is defined by a symplectic structure. Thus, the advantage of our approach is that it remains close to the standard algebraic setting of QFT. The obvious disadvantage is the restriction to formal power series in the deformation parameter. This, in particular, does not allow us to study nonperturbative NC effects, such as causality violation in the field propagation. The main physical motivation for our investigations is to make contact to NC cosmology, in particular to perturbative NC effects in the cosmic microwave background, and to NC black hole physics. The NC gravity solutions recently obtained in [10,11,12] provide a natural application for the methods developed in this paper.

The outline of this paper is as follows: In Section 2 we review the basics of twist-deformed differential geometry in the sense of [13,14]. Based on these methods, we construct action functionals and equation of motion operators for free massive scalar fields for a large class of triangular Drinfel'd twists. In Section 3 we construct the deformed Green's operators and the deformed fundamental solution for the type of wave operators, constructed in Section 2, in terms of formal power series. The deformed wave propagation is causal with respect to the undeformed metric field. The deformed fundamental solution is used to canonically construct a symplectic vector space and to define $\ast$-algebras of field observables in Section 4. We conclude and give an outlook to possible generalizations and applications of our formalism in Section 5.

2 Twist-deformed differential geometry, scalar field action functionals and equation of motion operators

In this section we briefly describe how to construct action functionals and equation of motion operators for scalar fields on a large class of NC manifolds. We focus on the class of $\star$-products, which can be obtained by a Drinfel'd twist $\mathcal{F} \in U\Xi[[\lambda]] \otimes_{\mathbb{C}} U\Xi[[\lambda]]$, where $U\Xi$ is the universal enveloping algebra of the complexified vector fields on the manifold $\mathcal{M}$, λ is the deformation parameter and $[[\lambda]]$ denotes formal power series in λ. The Drinfel'd twist should be finitely-generated, i. e. $\mathcal{F}$ consists of sums of finite products of vec-

tor fields at every order in λ. For more information on these deformations see e. g. [14]. Given the commutative algebra of smooth complex-valued functions $\mathcal{A} = \big(C^\infty(\mathcal{M}), \cdot\big)$ on the manifold $\mathcal{M}$, we can deform it into an associative, but in general noncommutative, algebra $\mathcal{A}_\star = \big(C^\infty(\mathcal{M})[[\lambda]], \star\big)$ with the $\star$-product defined by

$$h \star k := \bar{f}^\alpha(h) \cdot \bar{f}_\alpha(k) \ , \ \text{for all } h, k \in \mathcal{A}_\star \ , \tag{1}$$

where $\bar{f}^\alpha \otimes \bar{f}_\alpha = \mathcal{F}^{-1}$ is the inverse twist which acts on functions via the Lie derivative. We restrict ourselves in the following to hermitian $\star$-products satisfying $(h \star k)^* = k^* \star h^*$, for all $h, k \in \mathcal{A}_\star$, where $*$ is the standard involution on $\mathcal{A}$.

In the same way we can deform the exterior algebra of complexified differential forms $\big(\Omega^\bullet, \wedge, d\big)$ on $\mathcal{M}$ into $\big(\Omega^\bullet[[\lambda]], \wedge_\star, d\big)$ with the deformed wedge-product defined by

$$\omega \wedge_\star \omega' := \bar{f}^\alpha(\omega) \wedge \bar{f}_\alpha(\omega') \ , \ \text{for all } \omega, \omega' \in \ \Omega^\bullet[[\lambda]] \ , \tag{2}$$

where the inverse twist again acts on differential forms via the Lie derivative. Note that, since Lie derivatives and exterior derivatives commute, the exterior differential can be chosen to be undeformed and satisfies the graded Leibniz rule

$$d\big(\omega \wedge_\star \omega'\big) = (d\omega) \wedge_\star \omega' + (-1)^{\deg(\omega)} \omega \wedge_\star (d\omega') \ , \ \text{for all } \omega, \omega' \in \Omega^\bullet[[\lambda]] \ . \tag{3}$$

Next, we consider the integration of differential forms on a twist-deformed manifold $\mathcal{M}$. Since, as vector spaces, the deformed and the formal power series of the undeformed differential forms are isomorphic, we can define integration in terms of the commutative integral. Identifying a volume form vol on $\mathcal{M}$ in general leads to a non-cyclic integral on $\mathcal{A}_\star$. Furthermore, it turns out that the integral for general $\mathcal{F}$ does not even possess the weaker property of "graded cyclicity" given by

$$\int_\mathcal{M} \omega \wedge_\star \omega' = (-1)^{\deg(\omega)\deg(\omega')} \int_\mathcal{M} \omega' \wedge_\star \omega = \int_\mathcal{M} \omega \wedge \omega' \ , \tag{4}$$

for all $\omega, \omega' \in \Omega^\bullet[[\lambda]]$ with $\deg(\omega) + \deg(\omega') = \dim(\mathcal{M})$ and $\mathrm{supp}(\omega) \cap \mathrm{supp}(\omega')$ compact [1]. One explicit example of a twist not satisfying graded cyclicity is the Jordanian twist $\mathcal{F} = \exp\big(\frac{1}{2} H \otimes \log(1 + \lambda E)\big)$ with $[H, E] = 2E$. Since graded cyclicity is a property which drastically simplifies the construction of equations of motion from a given action functional, we restrict ourselves in the following to a subclass of Drinfel'd twists in order to obtain this property. It can be checked that Drinfel'd twists satisfying $S(\bar{f}^\alpha) \cdot \bar{f}_\alpha = 1$ fulfil graded cyclicity [15]. Here S is the antipode on the universal enveloping algebra $U\Xi[[\lambda]]$, which is defined as an algebra antihomomorphism acting on vector fields $u \in \Xi[[\lambda]]$ by $S(u) = -u$ and $S(1) = 1$. Note that the condition demanded above is not

[1] Let $\omega := \sum \lambda^n \omega_{(n)} \in \Omega^\bullet[[\lambda]]$ and $\omega' := \sum \lambda^n \omega'_{(n)} \in \Omega^\bullet[[\lambda]]$. The statement $\mathrm{supp}(\omega) \cap \mathrm{supp}(\omega')$ compact is an abbreviation for $\mathrm{supp}(\omega_{(n)}) \cap \mathrm{supp}(\omega'_{(m)})$ compact for all $n, m \in \mathbb{N}^0$.

too restrictive, since it allows the whole class of abelian twists (also called Reshetikhin-Jambor-Sykora (RJS) twists [16,17]).

In order to construct kinetic terms of scalar field action functionals, we further require a deformed (pseudo-)hermitian structure $h_\star$ on one-forms. This structure can be defined using the inverse metric field of NC gravity $g^{-1} := g^{-1\alpha} \otimes_\star g_\alpha^{-1} \in \varXi[[\lambda]] \otimes_\star \varXi[[\lambda]]$ and the pairing $\langle \cdot, \cdot \rangle_\star$ between vector fields $\varXi[[\lambda]]$ and one-forms $\varOmega[[\lambda]]$ [13,14] by

$$h_\star(\omega, \omega') := \langle \langle \omega^*, g^{-1\alpha} \rangle_\star \star g_\alpha^{-1}, \omega' \rangle_\star \ , \ \text{for all } \omega, \omega' \in \varOmega[[\lambda]] \ . \tag{5}$$

Using the formalism described above, we are in the position to deform the standard quadratic scalar field action functional. We define

$$S_\star[\varPhi] := -\int_{\mathcal{M}} h_\star(d\varPhi, d\varPhi) \star \mathrm{vol}_\star - m^2 \int_{\mathcal{M}} \varPhi^* \star \varPhi \star \mathrm{vol}_\star \ , \tag{6}$$

where $\mathrm{vol}_\star \in \varOmega^\bullet[[\lambda]]$ is a real and nonvanishing top-form on $\mathcal{M}$ [2]. Due to graded cyclicity, hermiticity of $h_\star$ and reality of $\mathrm{vol}_\star$ we obtain $(S_\star[\varPhi])^* = S_\star[\varPhi]$.

The equation of motion for $\varPhi$ is obtained by varying the action (6). Since we are mainly interested in real scalar fields, we restrict ourselves from now on to this case. Using variations $\delta\varPhi$ of compact support and graded cyclicity we find

$$\delta S_\star[\varPhi] = \int_{\mathcal{M}} \delta\varPhi \cdot \tilde{P}_\star[\varPhi] \ , \tag{7}$$

with the $\mathbb{C}[[\lambda]]$-linear top-form valued differential operator $\tilde{P}_\star : C^\infty(\mathcal{M})[[\lambda]] \to \varOmega^\bullet[[\lambda]]$ given by

$$\tilde{P}_\star[\varphi] := \tilde{\Box}_\star[\varphi] + \big(\tilde{\Box}_\star[\varphi^*]\big)^* - m^2 \varphi \star \mathrm{vol}_\star - m^2 \mathrm{vol}_\star \star \varphi \ , \tag{8}$$

for all $\varphi \in C^\infty(\mathcal{M})[[\lambda]]$. The top-form valued d'Alembert operator $\tilde{\Box}_\star$ is defined by

$$\int_{\mathcal{M}} \psi^* \star \tilde{\Box}_\star[\varphi] := -\int_{\mathcal{M}} h_\star(d\psi, d\varphi) \star \mathrm{vol}_\star \ , \tag{9}$$

for all $\psi, \varphi \in C^\infty(\mathcal{M})[[\lambda]]$ with $\mathrm{supp}(\psi) \cap \mathrm{supp}(\varphi)$ compact. Demanding $\delta S_\star[\varPhi] = 0$ results in the top-form valued equation of motion $\tilde{P}_\star[\varPhi] = 0$. By defining $\tilde{P}_\star[\varphi] =: P_\star[\varphi] \star \mathrm{vol}_\star$, for all $\varphi \in C^\infty(\mathcal{M})[[\lambda]]$, we obtain equivalently the equation of motion $P_\star[\varPhi] = 0$, where $P_\star : C^\infty(\mathcal{M})[[\lambda]] \to C^\infty(\mathcal{M})[[\lambda]]$ is now a $\mathbb{C}[[\lambda]]$-linear scalar differential operator.

Note that $P_\star$ is formally self-adjoint with respect to the scalar product

$$(\psi, \varphi)_\star := \int_{\mathcal{M}} \psi^* \star \varphi \star \mathrm{vol}_\star \ . \tag{10}$$

[2] One consistent choice of $\mathrm{vol}_\star$ is the classical volume form associated to the NC metric field g, since it is an element of $\varOmega^\bullet[[\lambda]]$, coordinate independent, real and nonvanishing on $\mathcal{M}$. Since we do not want to exclude different choices of $\mathrm{vol}_\star$ in our work, we keep it unspecified and only impose the natural conditions of reality and nondegeneracy.

More precisely,

$$(\psi, P_\star[\varphi])_\star = (P_\star[\psi], \varphi)_\star \tag{11}$$

holds true for all $\psi, \varphi \in C^\infty(\mathcal{M})[[\lambda]]$ with $\mathrm{supp}(\psi) \cap \mathrm{supp}(\varphi)$ compact. Furthermore, in the case of a deformed Lorentzian manifold $(\mathcal{M}, \star, g)$, the operator $P_\star$ is a formal deformation of a normally hyperbolic operator acting on $C^\infty(\mathcal{M})$, namely the Klein-Gordon operator.

This shows that we can explicitly construct quadratic scalar field action functionals leading to equation of motion operators which are formally self-adjoint with respect to the scalar product (10) and further are deformations of normally hyperbolic operators in the Lorentzian case. The generalization to generic matter fields requires the deformation quantization of hermitian vector bundles. For proofs on existence and uniqueness of these structures within the framework of formal deformation quantization see [18]. The explicit construction of realizations in twist-deformed NC geometry is currently in preparation [19].

3 Formal solutions of deformed wave equations

In this section we consider formal deformations of normally hyperbolic differential operators acting on formal power series of functions $C^\infty(\mathcal{M})[[\lambda]]$ on a twist-deformed globally hyperbolic Lorentzian manifold $(\mathcal{M}, \star, g)$. This means that $(\mathcal{M}, g|_{\lambda\to 0})$ is globally hyperbolic, but we do not impose conditions on the quantum corrections of g. The physics which can be described within this framework are free scalar (quantum) field theories on a large class of curved NC manifolds.

Let $P_\star : C^\infty(\mathcal{M})[[\lambda]] \to C^\infty(\mathcal{M})[[\lambda]]$ be a $\mathbb{C}[[\lambda]]$-linear differential operator, which is a formal deformation of a normally hyperbolic operator. More precisely, $P_\star$ is defined by the formal power series

$$P_\star := \sum_{n=0}^{\infty} \lambda^n P_{(n)} \ , \tag{12}$$

where $P_{(0)}$ is a normally hyperbolic operator and $P_{(n)} : C^\infty(\mathcal{M})[[\lambda]] \to C_0^\infty(\mathcal{M})[[\lambda]]$ are the $\mathbb{C}[[\lambda]]$-linear "quantum corrections" for $n > 0$. We demand that $P_\star$ is formally self-adjoint with respect to the scalar product (10), i. e. (11) holds true. We have constructed nontrivial examples of such operators in Section 2, but we do not restrict ourselves to these examples in the following and work with the abstract definition.

Note that, in order to formally solve the dynamics governed by $P_\star$ we have to add one technical assumption. We assume that $P_{(n)}$ are differential operators which map to functions of compact support for $n > 0$. This is, similar to the interactions of compact support in perturbative QFT, a technical assumption required for the construction of the formal power series and not motivated by physics. The support condition on $P_{(n)}$ is in particular satisfied for all twists

constructed by compactly supported vector fields. In case of a noncompactly supported twist $\mathcal{F}$, an infrared (IR) regularization is in general required. For this, we might, most preferably, try to approximate $\mathcal{F}$ by compactly supported twists or define by hand the regularized operators $P^{\mathrm{reg}}_{(n)} := \chi_n\, P_{(n)}$, where $\chi_n \in C^\infty_0(\mathcal{M})$ are cutoff functions satisfying $\chi_n|_{\mathcal{O}} \equiv 1$ for a sufficiently large spacetime region $\mathcal{O}$. Although the global aspects of QFT are known to change due to IR regularization, it was argued in [20] that the local observables in (commutative) QFT remain unaffected by details of the IR regularization. Since our NC QFTs are formal deformations of commutative QFTs, we expect that, similar to [20], the local formal NC physics is properly described by our models.

The main aim of this section is to construct deformed retarded and advanced Green's operators

$$\Delta_{\star\pm} := \sum_{n=0}^{\infty} \lambda^n \Delta_{(n)\pm} : C^\infty_0(\mathcal{M})[[\lambda]] \to C^\infty(\mathcal{M})[[\lambda]] \tag{13}$$

satisfying suitable properties. Even though some of the results obtained in this section could be derived from general considerations in deformation theory[3], we perform an explicit construction of the deformed Green's operators in order to make the investigation of their properties in this article self contained.

Theorems on the existence and uniqueness of Green's operators for the classical problem $\lambda \to 0$ can be found in [7] and apply to the λ^0-order of our problem. In particular, it was shown in [7] that there exists an unique retarded and advanced Green's operator $\Delta_\pm =: \Delta_{(0)\pm} : C^\infty_0(\mathcal{M}) \to C^\infty(\mathcal{M})$ satisfying

$$P_{(0)} \circ \Delta_\pm = \mathrm{id}_{C^\infty_0(\mathcal{M})} \ , \tag{14a}$$

$$\Delta_\pm \circ P_{(0)}\big|_{C^\infty_0(\mathcal{M})} = \mathrm{id}_{C^\infty_0(\mathcal{M})} \ , \tag{14b}$$

$$\mathrm{supp}(\Delta_\pm[\varphi]) \subseteq J_\pm(\mathrm{supp}(\varphi)) \ , \quad \text{for all } \varphi \in C^\infty_0(\mathcal{M}) \ , \tag{14c}$$

where $J_\pm(A)$ is the causal future/past of a subset A measured with respect to the undeformed spacetime metric $g|_{\lambda\to 0}$.

We extend these results to the NC setting by the following

Theorem 1 *Let $(\mathcal{M}, \star, g)$ be a deformed time-oriented, connected, globally hyperbolic manifold and let $P_\star := \sum \lambda^n P_{(n)}$ be a formal deformation of a normally hyperbolic operator acting on $C^\infty(\mathcal{M})[[\lambda]]$. Furthermore, let $P_{(n)}$ be finite-order differential operators with $Im(P_{(n)}) \subseteq C^\infty_0(\mathcal{M})$ for $n > 0$. Then there exist unique deformed Green's operators $\Delta_{\star\pm} := \sum \lambda^n \Delta_{(n)\pm}$ satisfying*

$$P_\star \circ \Delta_{\star\pm} = id_{C^\infty_0(\mathcal{M})[[\lambda]]} \ , \tag{15a}$$

$$\Delta_{\star\pm} \circ P_\star\big|_{C^\infty_0(\mathcal{M})[[\lambda]]} = id_{C^\infty_0(\mathcal{M})[[\lambda]]} \ , \tag{15b}$$

$$supp(\Delta_{(n)\pm}[\varphi]) \subseteq J_\pm(supp(\varphi)) \ , \quad \textit{for all } n \in \mathbb{N}^0 \textit{ and } \varphi \in C^\infty_0(\mathcal{M}) \ , \tag{15c}$$

[3] We thank the anonymous referee for pointing this out to us.

where $J_\pm$ is the causal future/past with respect to the classical metric $g|_{\lambda\to 0}$. The explicit expressions for $\varDelta_{(n)\pm}$, $n>0$, read

$$\varDelta_{(n)\pm} = \sum_{k=1}^{n}\sum_{j_1=1}^{n}\cdots\sum_{j_k=1}^{n}(-1)^k\delta_{j_1+\cdots+j_k,n} \\ \varDelta_\pm\circ P_{(j_1)}\circ\varDelta_\pm\circ P_{(j_2)}\circ\cdots\circ\varDelta_\pm\circ P_{(j_k)}\circ\varDelta_\pm\ , \quad (16)$$

where $\delta_{n,m}$ is the Kronecker-delta.

This theorem is proven in the Appendix A.

Next, we study properties of the deformed fundamental solution defined by the $\mathbb{C}[[\lambda]]$-linear map

$$\varDelta_\star := \varDelta_{\star+}-\varDelta_{\star-} : C_0^\infty(\mathcal{M})[[\lambda]]\to C^\infty_{\mathrm{sc}}(\mathcal{M})[[\lambda]]\ , \quad (17)$$

where $C^\infty_{\mathrm{sc}}(\mathcal{M})$ are the functions of spatially compact support. The importance of the fundamental solution lies in the fact that the covariant Poisson bracket relations (i. e. the Peierls bracket relations) of classical field theory and the Weyl relations of QFT can be defined by using this map. We obtain the following

Theorem 2 *Let $(\mathcal{M},\star,g)$ be a deformed time-oriented, connected, globally hyperbolic manifold and let $P_\star$ and $\varDelta_{\star\pm}$ be as above. Then the sequence of $\mathbb{C}[[\lambda]]$-linear maps*

$$0\longrightarrow C_0^\infty(\mathcal{M})[[\lambda]]\xrightarrow{P_\star}C_0^\infty(\mathcal{M})[[\lambda]]\xrightarrow{\varDelta_\star}C^\infty_{\mathrm{sc}}(\mathcal{M})[[\lambda]]\xrightarrow{P_\star}C^\infty_{\mathrm{sc}}(\mathcal{M})[[\lambda]] \quad (18)$$

is a complex, which is exact everywhere.

The proof of this theorem can be obtained by extending the proof of Theorem 3.4.7. of [7] to formal power series. For completeness, we provide the proof in the Appendix B.

4 Deformed symplectic vector space and $*$-algebras of field observables

In the standard construction of the QFT of a free real scalar field on a commutative globally hyperbolic manifold with a normally hyperbolic operator P one defines a symplectic structure ω on the real vector space $V := C_0^\infty(\mathcal{M},\mathbb{R})/P[C_0^\infty(\mathcal{M},\mathbb{R})]$ by using the fundamental solution and the undeformed version of the scalar product (10). Using the symplectic vector space (V,ω) one then defines the associated Weyl algebra of field observables. For details on this construction see [7].

We now show that a similar construction is also possible in the twist-deformed case, leading to deformed $*$-algebras of field observables. Consider a real scalar field on a deformed globally hyperbolic manifold with an equation of motion given by a deformed normally hyperbolic operator $P_\star$, satisfying the

properties defined above. We further demand the reality condition of Section 2 given by

$$\big(\tilde{P}_\star[\varphi]\big)^* = \big(P_\star[\varphi]\star \mathrm{vol}_\star\big)^* = \tilde{P}_\star[\varphi^*]\ , \quad \text{for all } \varphi\in C^\infty(\mathcal{M})[[\lambda]]\ , \tag{19}$$

since it naturally emerges from an action principle.

Using Theorem 1 we obtain unique deformed Green's operators $\Delta_{\star\pm}$ and the fundamental solution $\Delta_\star = \Delta_{\star+}-\Delta_{\star-}$. We define the following map

$$\tilde{\omega}_\star : C_0^\infty(\mathcal{M})[[\lambda]]\times C_0^\infty(\mathcal{M})[[\lambda]]\to \mathbb{C}[[\lambda]],\ (\psi,\varphi)\mapsto \tilde{\omega}_\star(\psi,\varphi) = (\psi,\Delta_\star[\varphi])_\star\ , \tag{20}$$

which is the basic ingredient in the construction of the deformed symplectic vector space $V_\star$. In order to define $V_\star$, we first have to restrict $C_0^\infty(\mathcal{M})[[\lambda]]$ to a suitable real subspace. We define the real vector space

$$H := \{\varphi\in C_0^\infty(\mathcal{M})[[\lambda]] : (\Delta_{\star\pm}[\varphi])^* = \Delta_{\star\pm}[\varphi]\}\ . \tag{21}$$

This vector space turns out to be a natural generalization of the classical space $C_0^\infty(\mathcal{M},\mathbb{R})$ due to the following

Proposition 1 *Consider the vector space H defined above. Then the following statements hold true:*

1.) Let $\psi\in C^\infty_{\mathrm{sc}}(\mathcal{M},\mathbb{R})[[\lambda]]$ be a real solution of the wave equations given by $P_\star$. Then there is a $\varphi\in H$, such that $\psi=\Delta_\star[\varphi]$.
2.) The kernel of the fundamental solution $\Delta_\star$ restricted to H is given by $\mathrm{Ker}(\Delta_\star)|_H = P_\star[C_0^\infty(\mathcal{M},\mathbb{R})[[\lambda]]]$.
3.) Let $\varphi\in H$, then $(\varphi\star\mathrm{vol}_\star)^* = \varphi\star\mathrm{vol}_\star$.

This proposition is proven in the Appendix C.

We define the real factor space $V_\star := H/P_\star[C_0^\infty(\mathcal{M},\mathbb{R})[[\lambda]]]$. The map $\tilde{\omega}_\star$ (20) induces a map $\omega_\star$ on $V_\star$ by defining

$$\omega_\star : V_\star\otimes_{\mathbb{R}} V_\star \to \mathbb{R}[[\lambda]],\ ([\psi],[\varphi])\mapsto \tilde{\omega}_\star(\psi,\varphi) = (\psi,\Delta_\star[\varphi])_\star\ . \tag{22}$$

This map is well defined due to the anti-hermiticity property $(\psi,\Delta_\star[\varphi])_\star = -(\Delta_\star[\psi],\varphi)_\star$ for all $\psi,\varphi\in C_0^\infty(\mathcal{M})[[\lambda]]$, which follows from Lemma 2 (see Appendix B). Furthermore, $\omega_\star$ is nondegenerate due to Proposition 1, part 2.), and the nondegeneracy of the scalar product.

It remains to show the antisymmetry and reality of this map. Using Proposition 1, we obtain for all $\psi,\varphi\in H$

$$\begin{aligned}(\psi,\Delta_\star[\varphi])_\star &= \int_{\mathcal{M}} \psi^*\star\Delta_\star[\varphi]\star\mathrm{vol}_\star \overset{\mathrm{GC,RE}}{=} \int_{\mathcal{M}} (\psi\star\mathrm{vol}_\star)^*\star\Delta_\star[\varphi] \overset{3.)}{=} \\ \int_{\mathcal{M}} \psi\star\mathrm{vol}_\star\star\Delta_\star[\varphi] &\overset{\mathrm{GC}}{=} \int_{\mathcal{M}} \Delta_\star[\varphi]\star\psi\star\mathrm{vol}_\star \overset{\varphi\in H}{=} (\Delta_\star[\varphi],\psi)_\star \overset{\mathrm{AH}}{=} -(\varphi,\Delta_\star[\psi])_\star\ ,\end{aligned} \tag{23}$$

where we also have used graded cyclicity (GC), reality of $\mathrm{vol}_\star$ (RE) and anti-hermiticity of $\varDelta_\star$ (AH). From this identity we obtain that $\omega_\star$ is antisymmetric. Reality follows from

$$(\psi, \varDelta_\star[\varphi])^*_\star \overset{\text{HSP}}{=} (\varDelta_\star[\varphi], \psi)_\star \overset{\text{AH}}{=} -(\varphi, \varDelta_\star[\psi])_\star \overset{\text{AS}}{=} (\psi, \varDelta_\star[\varphi]) \ , \tag{24}$$

where we have used hermiticity of the scalar product (HSP), anti-hermiticity of $\varDelta_\star$ and antisymmetry of $\omega_\star$ (AS).

This shows that, given a twist-deformed real scalar field with equation of motion operator $P_\star$ defined as above, we can construct the symplectic vector space $(V_\star, \omega_\star)$. Using this vector space we can, guided by the commutative case [7], define an $*$-algebra of field observables as follows:

Definition 1 Let $(V_\star, \omega_\star)$ be a symplectic vector space. Let $\mathfrak{A}$ be an $*$-algebra over $\mathbb{C}[[\lambda]]$ with unit and let $W : V_\star \to \mathfrak{A}$ be a map, such that for all $\varphi, \psi \in V_\star$ we have

$$W(0) = 1 \ , \tag{25a}$$
$$W(-\varphi) = W(\varphi)^* \ , \tag{25b}$$
$$W(\varphi) \cdot W(\psi) = e^{-i\omega_\star(\varphi,\psi)/2} \ W(\varphi + \psi) \ . \tag{25c}$$

We call $\mathfrak{A}$ an $*$-algebra of Weyl-type, if it is generated by the elements $W(\varphi)$.

Note that, different to the commutative case, we did not demand $\mathfrak{A}$ to be a C^*-algebra, since we are considering algebras over $\mathbb{C}[[\lambda]]$ and not $\mathbb{C}$. See [21] for a review on $*$-algebras over ordered rings and their $*$-representation theory on pre-Hilbert spaces. It is well-known that the uniqueness (up to $*$-isomorphisms) of the Weyl algebra in commutative QFT strongly relies on the C^*-property. Thus, we expect a richer $*$-representation theory for the $*$-algebras of Weyl-type defined above. However, in case we would find a convergent deformation of the symplectic vector space $(V_\star, \omega_\star)$, what is strongly motivated by physics, we could define the Weyl system according to the conventional definition [7], including C^*-algebras.

A second possible definition of an $*$-algebra of field observables is the following:

Definition 2 Let $(V_\star, \omega_\star)$ be a symplectic vector space. Let $\mathfrak{A}$ be an $*$-algebra over $\mathbb{C}[[\lambda]]$ with unit and let $\varPhi : V_\star \to \mathfrak{A}$ be a $\mathbb{R}[[\lambda]]$-linear map, such that for all $\varphi, \psi \in V_\star$ we have

$$\varPhi(\varphi)^* = \varPhi(\varphi) \ , \tag{26a}$$
$$[\varPhi(\varphi), \varPhi(\psi)] = i\, \omega_\star(\varphi, \psi)\, 1 \ . \tag{26b}$$

We call $\mathfrak{A}$ an $*$-algebra of field polynomials, if it is generated by the elements 1 and $\varPhi(\varphi)$.

This definition of the $*$-algebra of field polynomials is closer to physics, since there the n-point correlation functions, i.e. expectation values of n-fold products of the linear field operators in some appropriate state, are of particular interest. This shows that there are natural definitions of field observable algebras in the NC setting, motivating the algebraic description of QFT on curved NC spacetimes. The explicit construction and investigation of these algebras is beyond the scope of our present work.

We conclude this section by briefly studying the observable algebras of a very special class of twist-deformations of the manifold $\mathcal{M}$. Let $(\mathcal{M}, g, \mathrm{vol})$ be a Lorentzian manifold with isometries given by a Lie algebra $\mathfrak{g}$ and with the volume form vol associated to g. Furthermore, let $\mathcal{F} \in U\mathfrak{g}[[\lambda]] \otimes_{\mathbb{C}} U\mathfrak{g}[[\lambda]] \subseteq U\Xi[[\lambda]] \otimes_{\mathbb{C}} U\Xi[[\lambda]]$. These Drinfel'd twists are called Killing twists, since they are constructed completely by Killing vector fields. It is easy to see that all $\star$-products drop out of the action (6) by using graded cyclicity and the $\mathfrak{g}$-invariance of vol and g. Thus, the equation of motion operator $P_\star = P$ is undeformed, as well as the fundamental solution $\Delta_\star = \Delta$. We further obtain that $V_\star = V$ and that the symplectic structure $\omega_\star = \omega$ is undeformed, leading to the undeformed observable algebras. This means that the free QFT on Killing deformed manifolds can not be distinguished from the commutative one. Of course, interacting QFTs on Killing deformed spacetimes will contain NC effects. Note that the Moyal-Weyl deformation of the Minkowski spacetime is a particular example of a Killing deformation, since the twist-generating vector fields $\{P_\mu := \partial_\mu\}$ are Killing.

5 Conclusions and outlook

In this paper we have made a first step towards generalizing the algebraic approach to QFT to the realm of formally deformed manifolds. The deformations we have considered are given by triangular Drinfel'd twists, together with the restriction $S(\bar{f}^\alpha) \cdot \bar{f}_\alpha = 1$, and include in particular all abelian (also called RJS) twists. We have constructed quadratic scalar field actions and equations of motion within the framework of twist-deformed differential geometry and have shown that these wave equations can be solved in terms of formal power series. More precisely, we have constructed deformed advanced and retarded Green's operators and the deformed fundamental solution explicitly up to all orders in the deformation parameter. The deformed wave propagation is, as expected, compatible with classical causality given by the metric field $g|_{\lambda\to 0}$, since we have considered formal deformations. Using the fundamental solution, we were able to construct a symplectic vector space and therewith define $*$-algebras of field observables for the NC quantum field, which in general depend on the deformation parameter λ. In the special case of the Moyal-Weyl deformed Minkowski spacetime, or more general for all Killing deformations of symmetric manifolds, the deformation parameter is absent in the observable algebra.

In future work it will be of particular importance to study the physics described by the $*$-algebras of field observables using explicit examples, e. g. cosmological models or black holes. For this purpose the NC gravity solutions recently obtained in [10, 11, 12] will be helpful. It would also be fruitful to study the differences to other approaches to NC QFT [3] within these models.

On the conceptual side, it would be interesting to include the ideas of locally covariant QFT [22] to the NC setting, see also [5] for such an attempt. If this is possible at all, and what is the role of the twisted diffeomorphisms [13, 14] within this approach, are issues left to future work.

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A Proof of Theorem 1

We perform a proof by induction. The zeroth order of (15) is assured by (14). Assume that we have constructed the Green's operators to order λ^{n-1}. In order λ^n (15a) reads

$$\sum_{m=0}^{n} P_{(m)} \circ \Delta_{(n-m)\pm} = 0 \ . \tag{27}$$

Let $\varphi \in C_0^\infty(\mathcal{M})$ be arbitrary. We can reformulate (27) into a Cauchy problem with respect to the classical operator $P_{(0)}$

$$P_{(0)}[\Delta_{(n)\pm}[\varphi]] = -\sum_{m=1}^{n} P_{(m)}[\Delta_{(n-m)\pm}[\varphi]] =: S \in C_0^\infty(\mathcal{M}) \ . \tag{28}$$

In order to satisfy (15c), we have to impose trivial Cauchy data of vanishing field and derivative on a Cauchy surface past/future to $\mathrm{supp}(S)$. The unique solution is

$$\Delta_{(n)\pm}[\varphi] = \Delta_\pm[S] = -\sum_{m=1}^{n} \Delta_\pm \circ P_{(m)} \circ \Delta_{(n-m)\pm}[\varphi] \ . \tag{29}$$

We obtain the support property for $n \geqq m > 0$

$$\begin{aligned}\mathrm{supp}(\Delta_\pm \circ P_{(m)} \circ \Delta_{(n-m)\pm}[\varphi]) &\subseteq J_\pm(\mathrm{supp}(P_{(m)} \circ \Delta_{(n-m)\pm}[\varphi])) \\ &\subseteq J_\pm(\mathrm{supp}(\Delta_{(n-m)\pm}[\varphi])) \subseteq J_\pm(J_\pm(\mathrm{supp}(\varphi))) \subseteq J_\pm(\mathrm{supp}(\varphi)) \ , \end{aligned} \tag{30}$$

where we have used that $P_{(m)}$ is a finite-order $\mathbb{C}$-linear differential operator, thus satisfying $\mathrm{supp}(P_{(m)}[\varphi]) \subseteq \mathrm{supp}(\varphi)$. This shows that (15c) is satisfied to order λ^n.

The equality of (29) and (16) can either be shown combinatorically, or by showing that (16) solves (15a) together with the support property (15c), and thus has to be equal to (29).

The remaining step is to prove the order λ^n of (15b). Plugging in the explicit form (16) one notices that every possible chain of operators, e. g.

$$\Delta_\pm \circ P_{(j_1)} \circ \Delta_\pm \circ \cdots \circ \Delta_\pm \circ P_{(j_k)} \ , \quad \text{with } j_1 + j_2 + \cdots + j_k = n \ , \tag{31}$$

occurs exactly twice in (15b), but with a different sign. Thus they cancel. □

B Proof of Theorem 2

In order to prove Theorem 2 we require the following two lemmas.

Lemma 1 *Let* $\varphi \in C_0^\infty(\mathcal{M})[[\lambda]]$*, then* $\Delta_\star[\varphi] \in C_{\mathrm{sc}}^\infty(\mathcal{M})[[\lambda]]$.

Proof Define $\psi = \sum \lambda^n \psi_{(n)} := \Delta_\star[\varphi]$. We obtain by using Theorem 1

$$\mathrm{supp}(\psi_{(n)}) \subseteq J_+(K^\varphi_{(n)}) \cup J_-(K^\varphi_{(n)}) \ , \quad \text{for all } n \in \mathbb{N}^0 \ , \tag{32}$$

where $K^\varphi_{(n)} := \bigcup_{m=0}^n \mathrm{supp}(\varphi_{(n-m)})$ is compact. Thus $\psi_{(n)} \in C_{\mathrm{sc}}^\infty(\mathcal{M})$ for all $n \in \mathbb{N}^0$. □

Lemma 2 $(\psi, \Delta_{\star\pm}[\varphi])_\star = (\Delta_{\star\mp}[\psi], \varphi)_\star$ *for all* $\psi, \varphi \in C_0^\infty(\mathcal{M})[[\lambda]]$.

Proof By Theorem 1 we have

$$(\psi, \Delta_{\star\pm}[\varphi])_\star = (P_\star \circ \Delta_{\star\mp}[\psi], \Delta_{\star\pm}[\varphi])_\star \ . \tag{33}$$

Using Theorem 1 and global hyperbolicity, one obtains that $\mathrm{supp}(\Delta_{\star\mp}[\psi]) \cap \mathrm{supp}(\Delta_{\star\pm}[\varphi])$ is compact for all $\psi, \varphi \in C_0^\infty(\mathcal{M})[[\lambda]]$. To finish the proof we use that $P_\star$ is formally self-adjoint. □

We now give a proof of Theorem 2.

Proof The sequence of maps forms a complex due to Theorem 1 and Lemma 1.

To prove the first exactness, let $\varphi \in C_0^\infty(\mathcal{M})[[\lambda]]$ such that $P_\star[\varphi] = 0$. Then $\varphi = \Delta_{\star\pm} \circ P_\star[\varphi] = 0$.

To prove the second exactness, let $\varphi \in C_0^\infty(\mathcal{M})[[\lambda]]$ such that $\Delta_\star[\varphi] = 0$. We define $\psi := \Delta_{\star\pm}[\varphi]$ and obtain using Theorem 1 and global hyperbolicity of $(\mathcal{M}, g|_{\lambda\to 0})$ that $\psi \in C_0^\infty(\mathcal{M})[[\lambda]]$. We find $P_\star[\psi] = P_\star \circ \Delta_{\star\pm}[\varphi] = \varphi$.

To prove the third exactness, let $\varphi = \sum \lambda^n \varphi_{(n)} \in C_{\mathrm{sc}}^\infty(\mathcal{M})[[\lambda]]$ such that $P_\star[\varphi] = 0$. We can find a family of compact sets $\{K_{(n)} : n \in \mathbb{N}^0\}$, such that $\mathrm{supp}(\varphi_{(n)}) \subseteq I_+(K_{(n)}) \cup I_-(K_{(n)})$, where $I_\pm(A)$ is the chronological future/past of a subset A with respect to the classical metric $g|_{\lambda\to 0}$. We decompose analogously to [7] $\varphi_{(n)} = \varphi^+_{(n)} + \varphi^-_{(n)}$, where $\mathrm{supp}(\varphi^\pm_{(n)}) \subseteq I_\pm(K_{(n)}) \subseteq J_\pm(K_{(n)})$. We define $\varphi^\pm := \sum \lambda^n \varphi^\pm_{(n)}$ and $\psi := \pm P_\star[\varphi^\pm]$. Using the support properties of $\varphi^\pm$ and global hyperbolicity, one obtains that $\psi \in C_0^\infty(\mathcal{M})[[\lambda]]$. To show that $\Delta_{\star\pm}[\psi] = \pm\varphi^\pm$, let $\chi \in C_0^\infty(\mathcal{M})[[\lambda]]$ be arbitrary. We obtain

$$(\chi, \Delta_{\star\pm}[\psi])_\star = (\Delta_{\star\mp}[\chi], \psi)_\star = \pm(\Delta_{\star\mp}[\chi], P_\star[\varphi^\pm])_\star = \pm(P_\star \circ \Delta_{\star\mp}[\chi], \varphi^\pm)_\star = (\chi, \pm\varphi^\pm)_\star \ , \tag{34}$$

where we have used Lemma 2, Theorem 1 and that $P_\star$ is formally self-adjoint. This shows that $\Delta_\star[\psi] = \varphi$. □

C Proof of Proposition 1

Proof of 1.):
Let $\psi \in C_{\mathrm{sc}}^\infty(\mathcal{M}, \mathbb{R})[[\lambda]]$ be a real solution satisfying $P_\star[\psi] = 0$. By Theorem 2 we know that there is a $\varphi \in C_0^\infty(\mathcal{M})[[\lambda]]$, such that $\psi = \Delta_\star[\varphi]$. From the reality of ψ we obtain

$$\big(\Delta_{\star+}[\varphi]\big)^* - \Delta_{\star+}[\varphi] = \big(\Delta_{\star-}[\varphi]\big)^* - \Delta_{\star-}[\varphi] =: 2\delta \in C_0^\infty(\mathcal{M})[[\lambda]] \ . \tag{35}$$

One obtains that δ is of compact support by using Theorem 1 and global hyperbolicity. Using $\delta^* = -\delta$ and $\delta = \Delta_{\star\pm} \circ P_\star[\delta]$ (see Theorem 1) we find that

$$\left(\Delta_{\star\pm}\left[\varphi + P_\star[\delta]\right]\right)^* = \Delta_{\star\pm}\left[\varphi + P_\star[\delta]\right] \ . \tag{36}$$

Thus $\varphi + P_\star[\delta] \in H$ with $\Delta_\star[\varphi + P_\star[\delta]] = \Delta_\star[\varphi] = \psi$. □

Proof of 2.):
Let $\varphi \in P_\star[C_0^\infty(\mathcal{M},\mathbb{R})[[\lambda]]]$, then there is a $\chi \in C_0^\infty(\mathcal{M},\mathbb{R})[[\lambda]]$, such that $\varphi = P_\star[\chi]$. We obtain

$$(\Delta_{\star\pm}[\varphi])^* = \chi^* = \chi = \Delta_{\star\pm}[\varphi] \ . \tag{37}$$

Thus $\varphi \in H$ and $\Delta_\star[\varphi] = \Delta_\star[P_\star[\chi]] = 0$.
Let now $\varphi \in H$ such that $\Delta_\star[\varphi] = 0$. Then by Theorem 2 there exists $\chi \in C_0^\infty(\mathcal{M})[[\lambda]]$, such that $\varphi = P_\star[\chi]$. Using the definition of H we obtain

$$0 = (\Delta_{\star\pm}[\varphi])^* - \Delta_{\star\pm}[\varphi] = \chi^* - \chi \ , \tag{38}$$

thus $\chi \in C_0^\infty(\mathcal{M},\mathbb{R})[[\lambda]]$. □

Proof of 3.):
Let $\varphi \in H$. Using reality of the top-form valued equation of motion operator (19) we obtain

$$(\varphi \star \mathrm{vol}_\star)^* = \left(\tilde{P}_\star\left[\Delta_{\star\pm}\left[\varphi\right]\right]\right)^* = \tilde{P}_\star\left[(\Delta_{\star\pm}\left[\varphi\right])^*\right] = \tilde{P}_\star\left[\Delta_{\star\pm}\left[\varphi\right]\right] = \varphi \star \mathrm{vol}_\star \ . \tag{39}$$

□

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